A FUNCTIONAL METHOD FOR LINEAR SETS, II.

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ABSTRACT

Various exceptional sets (such as Helson sets and sets of multiplicity) are constructed by means of a functional method using differentiable mappings.

§1. In this note we prove theorems related to the one following:

Let E be a compact set in $(-\infty, \infty)$ and suppose that for a sequence $\lambda_n \to +\infty$

$$e(\lambda_n x) \to 1 \text{ uniformly in } E \text{ (where } e(t) \equiv e^{2\pi i t}\text{)}.$$

Then there exists a function ϕ of class $C^\infty(-\infty, \infty)$ so that $\phi' > 0$, and $\phi(E)$ is a Kronecker set.

For definitions, and examples of Kronecker sets constructed by diverse techniques see [2, 4, 5, 8].

It is also proved that the somewhat stronger conditions $\phi \in C^2$, $\phi > 0$, $\phi'' > 0$ cannot be attained for certain sets E of the type prescribed (called "Dirichlet sets" in [2]). A related negative example is given in [2].

Applying similar arguments to a class of sets close to the one above, we construct Helson sets [3, 5] of a new type.

§2. Let E be interior to a compact interval I and $C^\infty(I)$ the usual linear metric space of smooth real-valued functions on I. We operate for the most part in the subspace $C^\infty_* = C^\infty_*(E; I) \subseteq C^\infty(I)$ of functions ϕ such that $\phi' = 1$ on E and $\phi^{[n]} = 0$ on E for $n \geqq 2$. (If E is perfect only the requirement on ϕ' need be stated.) The technical argument for the theorem stated in §1 is accomplished below; the passage to that theorem is then exactly as in [4]. A convenient abbreviation is $\|f\|_S \equiv \sup |f(x)|$, $x \in S$.

THEOREM 1. *Let the compact set E and the sequence (λ_n) be as in §1 and let h be a continuous unimodular function on E. Then for each element ϕ_0 in C^∞_* there is a sequence (ϕ_n) in C^∞_* such that*

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$$\| e(\lambda_n\phi_n) - h \|_E \to 0, \ \| \phi_n - \phi_0 \|_{C^\infty} \to 0.$$

Proof. We construct a sequence (ϕ_n) convergent to ϕ_0 in any fixed C^r-norm $(r \geqq 1)$. First, ϕ_0' can be approximated in C^r by a function ψ', equal to 1 on a neighborhood of E. This is the "C^r-synthesis of ϕ_0'" and is possible by [7]; the latter work contains a very deep theorem on synthesis. Integration of ψ' yields a function ψ close to ϕ_0 in C^r-norm, while $\psi' = 1$ on a neighborhood of E. Now E has Lebesgue measure 0, so that in the neighborhood of E we can find a covering by a disjoint closed intervals $I_s = [a_s, b_s]$ $(1 \leqq s \leqq N)$. Because $\psi' = 1$ on each I_s we have

$$|e(\lambda_n\psi(x)) - e(\lambda_n\psi(a_s))| = |e(\lambda_n x) - e(\lambda_n a_s)|, \text{ for } x \in I_s.$$

Hence $e(\lambda_n\psi) - e(\lambda_n\psi(a_s)) \to 0$ uniformly on $E \cap I_s$. Now let ψ_n be chosen so that $\psi_n' = 0$ on each I_s, while $e(\lambda_n\psi_n(a_s))e(\lambda_n\psi(a_s)) = h(a_s)$. Because $\lambda_n \to +\infty$ this can be achieved while $\psi_n \to 0$ in C^r.

Now $\psi + \psi_n$ belongs to C^∞_* while

$$\limsup \| e(\lambda_n\psi + \lambda_n\psi_n) - h \|_{E \cap I_s} \leqq \| h(a_s) - h \|_{E \cap I_s}.$$

The norms on the left decrease to 0 with $\max(b_s - a_s)$, and we obtain uniform approximation to h.

§3. Let $r > 2$ be an integer and $H = H_r$ be the set of sums $\sum_{k=1}^{\infty} \varepsilon_k r^{-k}$, $\varepsilon_k = 0$ or 1. For $r \geqq 5$ there is a good analogue of Theorem 1; in general, however, we are able only to obtain a theorem about transforms of sets H_r in the Euclidean plane R^2. (In fact the trick involved for $r \geqq 5$ is less interesting than the more difficult approach necessary for $r = 3, 4$.) In this connection see the work of Piateckii-Sapiro and Salem and Zygmund [3; V, VI].

THEOREM 2. *Let $r \geqq 5$. Then, excepting a set of first category in $C^\infty_*(H; I)$, each function transforms H onto a Helson set.*

Proof. Let h be a continuous unimodular function on H, and $\phi_0 \in C^\infty_*$. We claim that there is a sequence (ϕ_n) in C^∞_* so that

$$\| \phi_n - \phi_0 \|_{C^\infty} \to 0, \limsup \| e(r^n\phi_n) - h \|_H \leqq 2\sin\left(\frac{\pi}{8}\right) < 1.$$

We proceed as in the proof of Theorem 1 and find again intervals I_s. If n is sufficiently large then each interval I_s actually meeting H contains a point

$$x_s = \sum_1^n \varepsilon_k r^{-k} + \frac{1}{2} \sum_{n+1}^{\infty} r^{-k}.$$

Here x_s need not belong to H. As before, we can modify ϕ_0 to a function ϕ so that $e(r^n\phi(x_s))$ is any complex number of modulus 1 — to be specified in a moment. Suppose now that

$$x \in I_s \cap H, \text{ say } x = \sum_1^\infty \varepsilon_k' r^{-k}. \text{ Then}$$

$$\phi(x) = \phi(x_s) + \sum_1^n (\varepsilon_k' - \varepsilon_k) r^{-k} + \sum_{n+1}^\infty \left(\varepsilon_k' - \frac{1}{2}\right) r^{-k},$$

$$r^n\phi(x) = r^n\phi(x_s) + u + w,$$

where $u \in Z$ and $|w| \leqq \frac{1}{2}(r-1)^{-1} \leqq \frac{1}{8}$. From the formula $|e(t) - 1)| = 2|\sin \pi t|$, we find for $x \in I_s \cap H$

$$|e(r^n\phi(x)) - h(x)| \leqq |e(r^n\phi(x_s)) - h(x)| + 2\sin\frac{\pi}{8},.$$

Now the first term on the right can be made uniformly small by refinement of the intervals. I_s and by appropriate choice of $r^n\phi(x_s)$ (modulo 1); thus the claim is verified.

Using Baire's Theorem as in [4] we conclude that for all ϕ_0, excepting a set of first category,

$$\liminf_{n\to\infty} \| e(r^n\phi_0) - h \|_H \leqq 2\sin\frac{\pi}{8},$$

for *every* continuous unimodular h. To prove that then $\phi(H)$ is a Helson set, let μ be a complex measure in H.

$$\sup\left|\int e(r_n\phi)d\mu\right| \geqq \sup\left|\int hd\mu\right| - .6\|\mu\|$$
$$= .4\|\mu\|.$$

See [5, p. 115]. That completes the proof.

When $r = 3,4$ the device breaks down, although the theorem might hold for these values.

THEOREM 3. *Let $r \geqq 3$. Then, excepting a set of the first category, each mapping*

$$x \to (\phi_0(x), x), \quad \phi_0 \in C_*^\infty(H; I)$$

transforms H onto a Helson set in R^2.

Proof. For $n = 1,2,3,\cdots$, let $H_n^{\ 1},\cdots,H_n^{\ 4}$ be the closed subsets of H determined by $\varepsilon_{n+1},\varepsilon_{n+2}$. For a moment we focus on $H_n^{\ 1}\cap I_s$. In the proof of Theorem 3, let x_s be replaced by some $y_s^{\ 1}\in H_n^{\ 1}\cap I$. Constructing ϕ as before and taking any $y\in H_n^{\ 1}\cap I_s$ we have

$$r^n\phi(y) = r^n\phi(y_s^{\ 1}) + u + \sum_{n+3}^{\infty}\delta_k r^{n-k},$$

where $u\in Z$ and $\delta_k = -1,0,1$. Thus $\left|\sum_{n+3}^{\infty}\delta_k r^{n-k}\right| \leqq (r-1)^{-1}r^{-2} \leqq 1/18$. In this way we construct functions $\phi_n^1,\cdots,\phi_n^4$ so that $e(r^n\phi_n^{\ 1}),\cdots,e(r^n\phi_n^{\ 4})$ give a good approximation to H on $H_n^{\ 1},\cdots,H_n^{\ 4}$ respectively.

Observe that if $x\in H_n^{\ i}$ and $y\in H_n^{\ i}$ with $1\leqq i<j\leqq 4$,

$$r^nx - r^ny = u + v \text{ with } u\in Z \text{ and } \frac{1}{2}r^{-2} \leqq |v| \leqq (r-1)^{-1}.$$

But this means that there are functions $F_n^{\ 1},\cdots,F_n^{\ 4}$, periodic in R^2 and bounded in C^2-norm by some constant M_r, so that

$$F_n^{\ i}(r^nx) = \delta_{i,j} \text{ on } H_n^{\ j}\ (1\leqq i,j\leqq 4).$$

Again we apply Baire's Theorem to C_*^∞, but not *via* the norm-separability of the Banach space $C(H)$. Instead, let $\{\sigma\}$ be a sequence of complex measures in H of norm 1, whose weak*-closed convex hull is the unit ball in the space of measures. Let V_σ be the open subset of C_*^∞ defined as follows:

There exists a periodic function $F(x_1,x_2)$ in R^2, such that $\|F\|_{C^2}\leqq M_r$ and

$$\left|\int_H F(\phi_0(x),x)d\sigma\right| \geqq \frac{1}{4}\left(1-3\sin\frac{\pi}{18}\right).$$

This condition is certainly fulfilled if there is a set $T\subseteq H$ such that

$$|\sigma|(T) \geqq \frac{1}{4}, F(\phi_0,x) = 0 \text{ on } H\sim T \text{ and}$$

$$\int_T \left| F - \frac{d\sigma}{d|\sigma|}\right| |d\sigma| \leqq |\sigma|(T)\cdot 3\sin\frac{\pi}{18}.$$

Observe that the density of each V_σ is ensured by the construction given before; as $H = \bigcup_1^4 H_n^{\ i}$, so $\cap V_\sigma$ is a dense G_δ in C_*^∞. But that intersection contains only functions ϕ_0 with the required properties.

It seems likely that similar theorems are true for the PV numbers > 2 [6], provided the plane is replaced by a space of large dimension.

§4. In this paragraph the introduction of the spaces C_*^∞ is justified by a negative result for $C^2(I)$. Let B be a sequence of positive integers containing arbitrarily long sequences of consecutive integers, and let F be the set of sums $\sum^*\varepsilon_k 2^{-k}$ where $\varepsilon_k = 0$ or 1 and $\sum^*$ means that $\varepsilon_k = 0$ when $k \in B$. Plainly F has the property required of E in §1, and the condition on B is compatible with a technical condition:

For each K the inequality $|b_1 - 2b_2| < K$ $(b_i \in B)$ has only a finite number of solutions.

Let μ be the *Lebesgue* measure on F, the usual product measure. Observe that if u and v are positive integers and $[u,v] \cap B = \emptyset$ then F contains all sums $\sum_u^v \varepsilon_k 2^{-k}$: an arithmetic progression of 2^{v-u+1} terms and difference 2^{-v}. Let ν be the probability distributed uniformly on this progression; then ν is a *factor* of μ: for a probability ν' in F we can write $\mu = \nu * \nu'$. Hence, for any continuous function ϕ on F

$$\left|\int e(\lambda\phi)d\mu\right| \leqq \sup\left|\int e(\lambda\phi(x+y))\nu(dy)\right|, \qquad x \in F.$$

THEOREM 4. *Let $\phi \in C^2(I)$ and $\phi' > 0$, $\phi'' > 0$. Then the measure ν can be chosen, as a function of $\lambda > 0$, so that the supremum on the right tends to 0 as $\lambda \to +\infty$. In particular $\phi(F)$ is a set of multiplicity in the narrow sense.*

Proof. The argument is suggested by Weyl's criterion for uniform distribution and an inequality of van der Corput [1, pp. 71–73] on exponential sums. Using the technical condition on B we can associate to each $\lambda > \lambda_0$ an interval $[u,v]$ so that

(i) $[u,v] \cap B = \emptyset$, $u \to +\infty$, $v - u \to +\infty$ as $\lambda \to +\infty$, and either

(ii) $u + 2v = 3[\log\lambda/\log 2]$, or

(iii) $u + 2v = \frac{3}{2}[\log\lambda/\log 2]$.

We handle (ii) and (iii) in order.

(ii) By Taylor's theorem, for $0 \leqq m < 2^{v-u+1}$,

$$\begin{aligned}\lambda\phi(x + m2^{-v}) &= \lambda\phi(x) + \lambda\phi'(x)2^{-v}m + \lambda O(m^2 4^{-v}) \\ &= \lambda\phi(x) + \lambda\phi'(x)2^{-v}m + o(1).\end{aligned}$$

Now $\lambda 2^{-v} = o(1)$ and $\lambda 2^{-v} \cdot 2^{v-u} \to +\infty$ so the exponential sum is easily shown to be $o(2^{v-u})$.

(iii) By the inequality of [1, p. 71] it is enough to prove that for each fixed $h \geqq 1$, uniformly in x,

$$\sum_{m=0}^{2^{v-u+1}} e(\lambda\phi(x+m2^{-v}))e(-\lambda\phi(x+m2^{-v}+h2^{-v})) = o(2^{v-u}).$$

For this purpose let D_ξ denote differentiation with respect to ξ. By the mean-value theorem

$$\begin{aligned} &D_\xi[\lambda\phi(x+\xi 2^{-v}) - \lambda\phi(x+\xi 2^{-v}+h2^{-v})] \\ &= -\lambda 4^{-v}h\phi''(x+\xi 2^{-v}+\theta h 2^{-v}), \text{ with } 0<\theta<1, \\ &= -\lambda 4^{-v}h(\phi''(x)+o(1)), \text{ when } 0 \leqq \xi \leqq 2^{v-u+1}. \end{aligned}$$

Here $\lambda 4^{-v} = o(1)$ while $\lambda 4^{-v}2^{v-u} = \lambda 2^{-v-u} \to +\infty$ because $-v-u+\frac{2}{3}u+\frac{4}{3}v$ tends to $+\infty$. Thus the bound $o(2^{v-u})$ is obtained by geometrical reasoning on the distribution (modulo 1) of the sequence

$$\lambda\phi(x+m2^{-v}) - \lambda\phi(x+m2^{-v}+h2^{-v}), \qquad 0 \leqq m < 2^{v-u+1}.$$

This proves Theorem 4.

It is known that a Helson set is a set of uniqueness for Fourier-Stieltjes transforms [5, p. 119] so that Theorem 4 is indeed a complement to the others. It would interesting to find conditions on ϕ strong enough to yield

$$\int e(\lambda\phi)d\mu = O(|\lambda|^{-\alpha}) \text{ for some } \alpha > 0.$$

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